Roll No. [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] **Total No. of Pages : 03**

Total No. of Questions : 07

M.Sc Mathematics (2018 Batch) (Sem.–1)

ORDINARY DIFFERENTIAL EQUATIONS AND SPECIAL FUNCTIONS

Subject Code : MSM-104-18

Paper ID : [75132]

Time : 3 Hrs. **Max. Marks : 70**

INSTRUCTION TO CANDIDATES :

1. **SECTION-A is COMPULSORY consisting of FIVE questions carrying TWO marks each.**
2. **SECTION - B & C. have THREE questions each.**
3. **Attempt any FOUR questions from SECTION B & C carrying FIFTEEN marks each.**
4. **Select atleast TWO questions from SECTION - B & C each.**

SECTION-A

1. Answer briefly :

a. Convert the differential equation $y''(t) = 3y'(t) + 8y(t) - 5y^2(t)$ in to a system of first order differential equations.

b. Convert the following Bessel's equation $x^2y'' + xy' + (x^2 - \alpha^2)y = 0$ in to standard Sturm-Liouville form.

c. Find a minimum value for the radius of convergence of a power series solution of

$$(x + 1)y'' - 3xy' + 2y = 0$$

about $x_0 = 1$.

d. Express $f(x) = x^4 + 2x^3 - 6x^2 + 5x - 3$ in terms of Legendre polynomials.

e. State Picard's Existence Theorem for a system of differential equations in n unknowns.

SECTION-B

2. (a) Consider the linear system $\frac{dx}{dt} = A\,x\,(t)$, with $A = \begin{bmatrix} -6 & -2 \\ 3 & 1 \end{bmatrix}$. (8)

Find the general solution and hence find the solution with initial condition $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. What is $\lim_{t \to \infty} x(t)$ in this case?

(b) Consider the third-order differential equation : (7)

$$\frac{d^3y}{dx^3} = x^2 + y\frac{dy}{dx} + \left(\frac{d^2y}{dx^2}\right)^2.$$

Does there exist a unique solution ϕ of the given equation such that

$$\phi(0) = 1, \phi'(0) = -3, \phi''(0) = 0?$$

Explain precisely why or why not.

3. (a) Use the operator method to find the general solution of the following linear system : (8)

$$\frac{dx}{dt} + \frac{dy}{dt} - x = -2t$$

$$\frac{dx}{dt} + \frac{dy}{dt} - 3x - y = t^2.$$

(b) Determine the nature of the critical point (0, 0) of the system : (7)

$$\frac{dx}{dt} = 2x - 7y,$$

$$\frac{dy}{dt} = 3x - 8y$$

and determine whether or not the point is stable.

4. (a) Prove that the eigen values of the following regular Sturm-Liouville problem (7)

$$-(p(x)y')' + q(x)y = \lambda r(x)y,\ \alpha_1 y(a) + \beta_1 y'(a) = 0,\ \alpha_2 y(b) + \beta_2 y'(b) = 0,$$

are real.

(b) Find eigen values and eigen functions for the Sturm-Liouville problem (8)

$$y'' + \lambda y = 0, \quad y'(0) = 0, y'(l) = 0,$$

where $l > 0$ is a constant and λ is a parameter.

SECTION-C

5. (a) Find at least the first four nonzero terms in a power series expansion of the solution to the given initial value problem. (8)

$$y'' - (\sin x)y = 0, y(\pi) = 1, y'(\pi) = 0.$$

(b) Use the method of Frobenius to find the solution near $x = 0$ of the following differential equation : (7)

$$2x\frac{d^2y}{dx^2} + \frac{dy}{dx} + 2y = 0.$$

6. (a) The Chebyshev differential equation is : (8)

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0,$$

Where α is *a* constant. Determine two solutions in powers of x for $|x| < 1$. Also show that if α is a nonnegative integer n, then there is a polynomial solution of degree n.

(b) Show that the Bessel's function of the first kind of order zero. $J_0(k\,x)$. Where k is a constant, satisfies the differential equation : (7)

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + k^2 xy = 0.$$

7. Consider the following Hermite equation :

$$y'' - 2xy' + \lambda y = 0, -\infty < x < \infty,$$

Where λ is a constant.

(a) Find the first four terms in each of two solutions about $x = 0$. (9)

(b) Show that if λ is a nonnegative even integer, then one or the other of the series solutions terminates and becomes a polynomial. Find the polynomial solutions for $\lambda = 0, 2, 4$ and 6. (3)

(c) The Hermite polynomial $H_n(x)$ is defined as the polynomial solution of the Hermite equation with $\lambda = 2n$ for which the coefficient of x_n is $2n$. Find $H_0(x)$,$H_1(x)$,$H_2(x)$, and $H_3(x)$. (3)